ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Sparse stochastic processes

Chapter 6
Splines and wavelets

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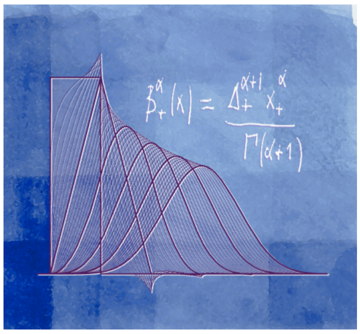
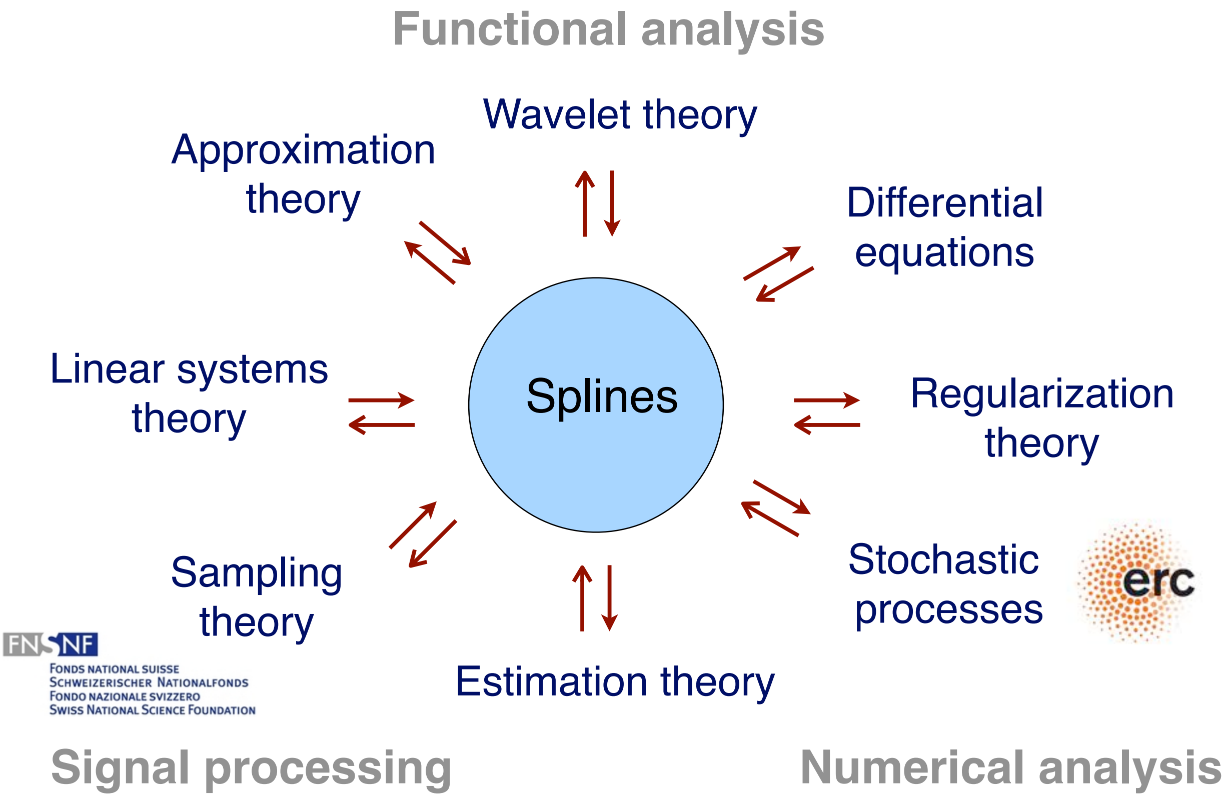


March 2017

EDEE Course

Splines as a unifying mathematical concept



CONTENT

Fourier-multiplier operators

Linear shift-invariant operator $\mathrm{L} : \mathcal{X}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ with $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{X}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$

- Frequency response: $\widehat{L}(\boldsymbol{\omega}) = \mathcal{F}\{\mathrm{L}\delta\}(\boldsymbol{\omega})$

  For all $\varphi \in \mathcal{X}(\mathbb{R}^d)$, $\quad \mathrm{L}\varphi(\boldsymbol{r}) = \mathcal{F}^{-1}\{\hat{\varphi}(\boldsymbol{\omega})\widehat{L}(\boldsymbol{\omega})\}(\boldsymbol{r})$

- Null space of L: $\quad \mathcal{N}_{\mathrm{L}} = \{p_0 \in \mathcal{X}(\mathbb{R}^d) : \mathrm{L}\{p_0\} = 0\}$

Proposition
If L is LSI and $p_{\boldsymbol{n}}(\boldsymbol{r}) = \mathrm{e}^{\langle \boldsymbol{z}_0, \boldsymbol{r}\rangle}\boldsymbol{r}^{\boldsymbol{n}} \in \mathcal{N}_{\mathrm{L}}$ with $\boldsymbol{z}_0 \in \mathbb{C}^d$, then $\mathcal{N}_{\mathrm{L}}$ does necessarily include all **exponential monomials** of the form $p_{\boldsymbol{m}}(\boldsymbol{r}) = \mathrm{e}^{\langle \boldsymbol{z}_0, \boldsymbol{r}\rangle}\boldsymbol{r}^{\boldsymbol{m}}$ with $\boldsymbol{m} \leq \boldsymbol{n}$. If $\mathcal{N}_{\mathrm{L}}$ is finite-dimensional, it can only consist of atoms of that particular form.

- Green's function of L (if it exists): $\quad \rho_{\mathrm{L}}$ such that $\mathrm{L}\rho_{\mathrm{L}} = \delta$

  Primary Green's function: $\quad \rho_{\mathrm{L}}(\boldsymbol{r}) = \mathcal{F}^{-1}\left\{\frac{1}{\widehat{L}(\boldsymbol{\omega})}\right\}(\boldsymbol{r})$

  Many equivalent Green's function: $\quad \rho_{\mathrm{L}} + p_0$ where $p_0 \in \mathcal{N}_{\mathrm{L}}$

Spline-admissible operators

Definition
The Fourier-multiplier operator $\mathrm{L}: \mathcal{X}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ with frequency response $\widehat{L}(\boldsymbol{\omega})$ is called **spline admissible** if $\rho_{\mathrm{L}} = \mathcal{F}^{-1}\{1/\widehat{L}(\boldsymbol{\omega})\}$ is an ordinary function of slow growth.

Definition
The Fourier-multiplier $\widehat{L}(\boldsymbol{\omega})$ is **of order** $\gamma \in \mathbb{R}^+$ if there exists $R \in \mathbb{R}^+$ and C such that

$$C|\boldsymbol{\omega}|^\gamma \le |\widehat{L}(\boldsymbol{\omega})|, \qquad \text{for all } |\boldsymbol{\omega}| \ge R \text{ where } \gamma \text{ is critical.}$$

- Example 1: D^γ with frequency response $(\mathrm{j}\omega)^\gamma$ is spline-admissible of order γ

$$\rho_{\mathrm{D}^\gamma}(r) = \mathcal{F}^{-1}\left\{\frac{1}{(\mathrm{j}\omega)^\gamma}\right\}(r) = \frac{r_+^{\gamma-1}}{\Gamma(\gamma)}$$

- Example 2: Partial differential operator $\mathrm{L}_N = \sum_{|\boldsymbol{n}|<N} a_{\boldsymbol{n}} \partial^{\boldsymbol{n}}$ with $a_{\boldsymbol{n}} \in \mathbb{R}^d$
  - Existence of Green's function follows from Malgrange-Ehrenpreis theorem
  - Operator is called ***elliptic*** if $\widehat{L}_N(\boldsymbol{\omega})$ vanishes at origin and nowhere else
  - If L_N is of order γ than it is also called ***quasi-elliptic***

Spline and operators

Definition
The function $s(\boldsymbol{r})$ (possibly of slow growth) is called a **cardinal** L**-spline**

$$\Leftrightarrow \qquad \mathrm{L}s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{r}-\boldsymbol{k})$$

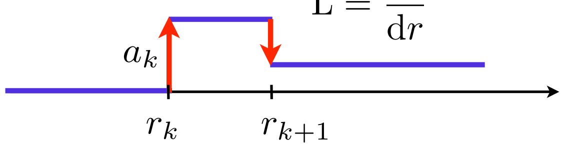
Definition
The function $s(\boldsymbol{r})$ (possibly of slow growth) is a **nonuniform** L**-spline** with knots $\{\boldsymbol{r}_k\}_{k\in S}$

$$\Leftrightarrow \qquad \mathrm{L}s(\boldsymbol{r}) = \sum_{k\in S} a_k \delta(\boldsymbol{r}-\boldsymbol{r}_k)$$

$$\Longrightarrow \qquad s(\boldsymbol{r}) = p_0(\boldsymbol{r}) + \sum_{k\in S} a_k \rho_{\mathrm{L}}(\boldsymbol{r}-\boldsymbol{r}_k)$$

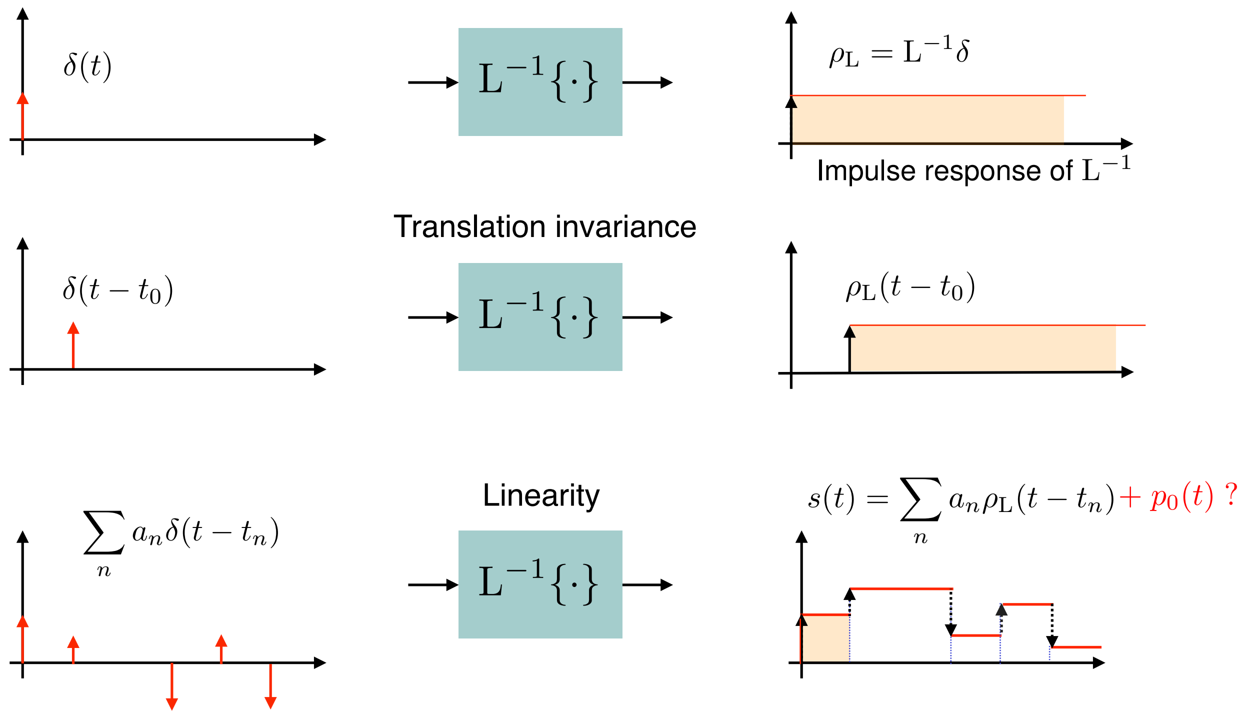
Spline theory: (Schultz-Varga, 1967; Jerome-Schumaker 1969)

$$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}r}$$



Innovation-based synthesis

$$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{D} \quad \Rightarrow \quad \mathrm{L}^{-1}\text{: integrator}$$



Riesz bases

Definition: A sequence of functions $\{\phi_k(\boldsymbol{r})\}_{k\in\mathbb{Z}}$ in $L_2(\mathbb{R}^d)$ forms a **Riesz basis** if and only if there exist two constants A and B such that

$$A\|c\|_{\ell_2} \leq \|\sum_{k\in\mathbb{Z}} c_k \phi_k(\boldsymbol{r})\|_{L_2(\mathbb{R}^d)} \leq B\|c\|_{\ell_2} \quad \text{for any sequence } c = (c_k) \in \ell_2.$$

- Continuous-discrete norm equivalence: $\|s\|_{L_2} \asymp \|c\|_{\ell_2}$
- Ensures stability of representation
- Implies linear independence of $\{\phi_k(\boldsymbol{r})\}_{k\in\mathbb{Z}}$
- Orthogonality $\Leftrightarrow$ $A = B = 1$ (Parseval's relation)

Theorem: Let $\phi(\boldsymbol{r}) \in L_2(\mathbb{R}^d)$ be a B-spline-like generator with Fourier transform $\hat{\phi}(\boldsymbol{\omega})$. Then, $\{\phi(\boldsymbol{r} - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ forms a Riesz basis with Riesz bounds A and B if and only if

$$0 < A^2 \leq \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |\hat{\phi}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2 \leq B^2 < \infty$$

Morever, the induced function space with $1 \leq p \leq +\infty$

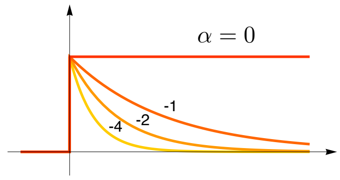
$$V_{\phi,p} = \left\{ s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\phi(\boldsymbol{r} - \boldsymbol{k}) : c \in \ell_p(\mathbb{Z}^d) \right\}$$

is a closed subspace of $L_p(\mathbb{R}^d)$ provided that $\sup_{\boldsymbol{r}\in[0,1]^d} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\phi(\boldsymbol{r} - \boldsymbol{k})| < +\infty$.

Construction of exponential B-spline

- Differential operator $\mathrm{L} = \mathrm{D} - \alpha \mathrm{Id} = \mathrm{P}_\alpha$

  Green's function: $\rho_\alpha(r) = \mathbb{1}_+(r)e^{\alpha r}$



- Discrete version of operator (weighted differences)

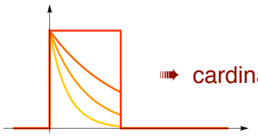
$$\mathrm{L_d}s(r) = \Delta_\alpha s(r) = s(r) - \mathrm{e}^\alpha s(r-1)$$

Fourier multiplier: $1 - \mathrm{e}^\alpha \mathrm{e}^{-\mathrm{j}\omega}$

- First-order exponential B-spline

$$\beta_\alpha(r) = \Delta_\alpha \rho_\alpha(r) = \mathbb{1}_{[0,1)}(r)\mathrm{e}^{\alpha r} \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1 - \mathrm{e}^\alpha \mathrm{e}^{-\mathrm{j}\omega}}{\mathrm{j}\omega - \alpha} = \frac{\widehat{L}_\mathrm{d}(\omega)}{\widehat{L}(\omega)}$$

pole



➠ cardinal (exponential) splines of minimal support

- Space of cardinal P_α-splines: $V_0 = \mathrm{span}\{\beta_\alpha(\cdot - k)\}_{k\in\mathbb{Z}}$

  β_α generates an *orthogonal* basis for the first-order cardinal exponential splines

6.4 GENERALIZED B-SPLINE BASIS

- B-spline properties
- B-spline factorization
- Polynomial B-splines
- Exponential B-splines
- Fractional B-splines
- *Additional brands of univariate B-splines*
- Multidimensional B-splines

Generalized B-spline

- Ingredients for B-spline construction
  - Green's function of operator L: ρ_{L}
  - Discrete version of operator: $\mathrm{L_d} s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} d_{\mathrm{L}}[\boldsymbol{k}] s(\boldsymbol{r}-\boldsymbol{k})$ with $d_{\mathrm{L}} \in \ell_1(\mathbb{Z}^d)$
  - Matching null-space constraint: $\mathrm{L_d} p_0(\boldsymbol{r}) = \mathrm{L} p_0(\boldsymbol{r}) = 0$ for all $p_0 \in \mathcal{N}_{\mathrm{L}}$

$$\beta_{\mathrm{L}}(\boldsymbol{r}) = \mathrm{L_d}\rho_{\mathrm{L}}(\boldsymbol{r}) = \mathcal{F}^{-1}\left\{\frac{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} d_{\mathrm{L}}[\boldsymbol{k}]\mathrm{e}^{-\mathrm{j}\langle\boldsymbol{k},\boldsymbol{\omega}\rangle}}{\widehat{L}(\boldsymbol{\omega})}\right\}(\boldsymbol{r})$$

Definition

The function β_{L} specified above is an admissible B-spline for L iff.

1) $\beta_{\mathrm{L}} \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$
2) it generates a Riesz basis of the space of cardinal L-splines.

- Generalized B-spline expansion

$s(\boldsymbol{r})$ is a cardinal L-spline $\quad\Leftrightarrow\quad s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\beta_{\mathrm{L}}(\boldsymbol{r}-\boldsymbol{k})$

B-spline properties

- Stable representation of cardinal L-splines

Formal definition: $V_{\mathrm{L}} = \left\{ s(\boldsymbol{r}) \in L_2(\mathbb{R}^d) : \mathrm{L}s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{r}-\boldsymbol{k}) \right\}$ (1)

$$\Longrightarrow \quad s(\boldsymbol{r}) = p_0(\boldsymbol{r}) + \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\rho_{\mathrm{L}}(\boldsymbol{r}-\boldsymbol{k})$$

B-spline representation: $V_{\mathrm{L}} = \left\{ s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\beta_{\mathrm{L}}(\boldsymbol{r}-\boldsymbol{k}) : c[\cdot] \in \ell_2(\mathbb{Z}^d) \right\}$ (2)

(2) $\Rightarrow$ (1): $\mathrm{L}s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\mathrm{L}\beta_{\mathrm{L}}(\boldsymbol{r}-\boldsymbol{k}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \underbrace{(c * d_{\mathrm{L}})[\boldsymbol{k}]}_{a[\boldsymbol{k}]}\,\delta(\boldsymbol{r}-\boldsymbol{k})$

(1) $\Rightarrow$ (2) = Completeness

Completeness of B-spline representation

- Reproduction of Green's functions

$$\mathrm{L}_{\mathrm{d}}^{-1}\delta(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} p[\boldsymbol{k}]\delta(\boldsymbol{r}-\boldsymbol{k}) = \mathcal{F}^{-1}\left\{\frac{1}{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} d_{\mathrm{L}}[\boldsymbol{k}]\mathrm{e}^{-\mathrm{j}\langle\boldsymbol{k},\boldsymbol{\omega}\rangle}}\right\}$$

where $p[\cdot]$ is a sequence of slow growth

$$\Longrightarrow \quad \sum_{\boldsymbol{k}\in\mathbb{Z}^d} p[\boldsymbol{k}]\beta_{\mathrm{L}}(\boldsymbol{r}-\boldsymbol{k}) = \mathrm{L}_{\mathrm{d}}^{-1}\beta_{\mathrm{L}}(\boldsymbol{r}) = \mathrm{L}_{\mathrm{d}}^{-1}\mathrm{L}_{\mathrm{d}}\rho_{\mathrm{L}}(\boldsymbol{r}) = \rho_{\mathrm{L}}(\boldsymbol{r})$$

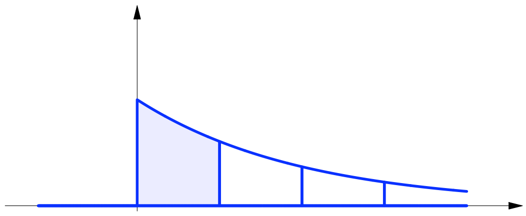
- Example: $\mathrm{L} = \mathrm{P}_\alpha$ with $\mathrm{Re}(\alpha) < 0$

$$\rho_\alpha(r) = \mathbb{1}_+(r)\mathrm{e}^{\alpha r} = \mathcal{F}^{-1}\left\{\frac{1}{\mathrm{j}\omega-\alpha}\right\}(r)$$

$$\hat{\Delta}_\alpha(\omega) = 1-\mathrm{e}^{\alpha-\mathrm{j}\omega}$$

$$p_\alpha[k] = \mathbb{1}_+[k]\mathrm{e}^{\alpha k} = \mathcal{F}_{\mathrm{d}}^{-1}\left\{\frac{1}{1-\mathrm{e}^{\alpha}\mathrm{e}^{-\mathrm{j}\omega}}\right\}[k]$$

$$\Longrightarrow \quad \mathbb{1}_+(r)\mathrm{e}^{\alpha r} = \sum_{k=0}^{\infty}\mathrm{e}^{\alpha k}\beta_\alpha(r-k)$$



Completeness (Cont'd)

$$\beta_{\mathrm{L}} \in L_1(\mathbb{R}^d) \quad \Rightarrow \quad \hat{\beta}_{\mathrm{L}}(\boldsymbol{\omega}) = \frac{\widehat{L}_{\mathrm{d}}(\boldsymbol{\omega})}{\widehat{L}(\boldsymbol{\omega})} \text{ is bounded and continuous}$$

$$\partial^{\boldsymbol{n}}\widehat{L}(\boldsymbol{0}) = 0, \text{ for all } \boldsymbol{n}\in\mathbb{N}^d, |\boldsymbol{n}|\le N \quad \Leftrightarrow \quad \mathcal{N}_{\mathrm{L}} \text{ includes all polynomials of degree } N$$

- Reproduction of null-space components
  - Consequence of null-space matching condition
  - Reproduction of polynomials ensured by Strang-Fix conditions of order N

$$\partial^{\boldsymbol{n}}\widehat{L}_{\mathrm{d}}(2\pi\boldsymbol{k}) = 0 \quad \Rightarrow \quad \begin{cases} \hat{\beta}_{\mathrm{L}}(\boldsymbol{0}) = 1, \\ \partial^{\boldsymbol{n}}\hat{\beta}_{\mathrm{L}}(2\pi\boldsymbol{k}) = 0, \end{cases} \quad \boldsymbol{k}\in\mathbb{Z}^d\backslash\{\boldsymbol{0}\}, \boldsymbol{n}\in\mathbb{N}^d \text{ with } |\boldsymbol{n}|\le N$$

  - Example: $\widehat{L}(\boldsymbol{0}) = 0 \quad \Rightarrow \quad \widehat{L}_{\mathrm{d}}(2\pi\boldsymbol{n}) = 0 \quad \Rightarrow$ reproduction of constant

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\beta_{\mathrm{L}}(\boldsymbol{r}-\boldsymbol{k}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d}\hat{\beta}_{\mathrm{L}}(2\pi\boldsymbol{n})\mathrm{e}^{\mathrm{j}\langle 2\pi\boldsymbol{n},\boldsymbol{r}\rangle} = \hat{\beta}_{\mathrm{L}}(\boldsymbol{0}) = 1 \quad \text{(Partition of unity)}$$

B-spline factorization

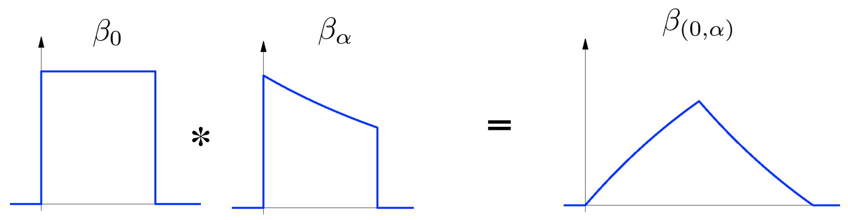
$$\mathrm{L} = \mathrm{L}_1\mathrm{L}_2 \quad \Rightarrow \quad \beta_{\mathrm{L}} = \beta_{\mathrm{L}_1} * \beta_{\mathrm{L}_2}$$

Proposition

Let $\beta_{\mathrm{L}_1}, \beta_{\mathrm{L}_2}$ be admissible B-splines for the operators L_1 and L_2, respectively. Then, $\beta_{\mathrm{L}} = \beta_{\mathrm{L}_1} * \beta_{\mathrm{L}_2}$ is an admissible B-spline for $\mathrm{L} = \mathrm{L}_1\mathrm{L}_2$ iff. there exists $A > 0$ such that

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^d} \left| \hat{\beta}_{\mathrm{L}_1}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})\hat{\beta}_{\mathrm{L}_2}(\boldsymbol{\omega} + 2\pi\boldsymbol{n}) \right|^2 \geq A > 0.$$

for all $\boldsymbol{\omega} \in [0, 2\pi]^d$. When $\mathrm{L}_1 = \mathrm{L}_2^{\gamma}$ for $\gamma \geq 0$, then the auxiliary condition is automatically satisfied.



$\mathrm{L}_1 = \mathrm{D} = \mathrm{P}_0$ $\quad$ $\mathrm{L}_2 = \mathrm{D} - \alpha\mathrm{Id} = \mathrm{P}_\alpha$ $\quad$ $\mathrm{L} = \mathrm{P}_0\mathrm{P}_\alpha = \mathrm{P}_{(0,\alpha)}$

Exponential B-splines

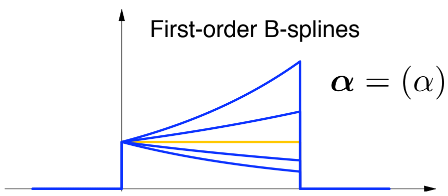
$\mathrm{L} = \mathrm{P}_{\boldsymbol{\alpha}} = \mathrm{P}_{\alpha_1} \cdots \mathrm{P}_{\alpha_N}$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$ and $\mathrm{P}_{\alpha_n} = \mathrm{D} - \alpha_n\mathrm{Id}$

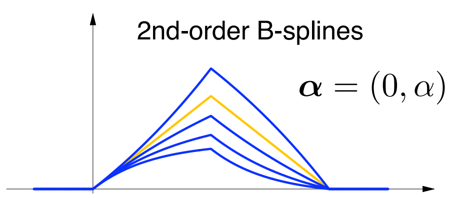
$\mathrm{L}_{\mathrm{d}} = \Delta_{\boldsymbol{\alpha}} = \Delta_{\alpha_1} \cdots \Delta_{\alpha_N}$ with $\Delta_{\alpha_n}\varphi(r) = \varphi(r) - \mathrm{e}^{\alpha_n}\varphi(r-1)$

$$\boxed{\begin{aligned} &\beta_\alpha(r) = \mathbb{1}_{[0,1)}\mathrm{e}^{\alpha r} \\ &\beta_{\boldsymbol{\alpha}}(r) = \left(\beta_{\alpha_1} * \beta_{\alpha_2} * \cdots * \beta_{\alpha_N}\right)(r) \end{aligned}} \quad \longleftrightarrow \quad \hat{\beta}_{\boldsymbol{\alpha}}(\omega) = \prod_{n=1}^{N} \frac{1 - \mathrm{e}^{\alpha_n}\mathrm{e}^{-\mathrm{j}\omega}}{\mathrm{j}\omega - \alpha_n}$$

poles

$\beta_{\boldsymbol{\alpha}}(r) = \Delta_{\boldsymbol{\alpha}}\rho_{\boldsymbol{\alpha}}(r)$ where $\rho_{\boldsymbol{\alpha}}(r)$: Green's function of $\mathrm{P}_{\boldsymbol{\alpha}}$





- Compactly supported in $[0, N]$; generate Riesz bases for cardinal $\mathrm{P}_{\boldsymbol{\alpha}}$-splines
- Piecewise exponential with maximal smoothness: Hölder-continuous of order $(N-1)$

Fractional B-splines

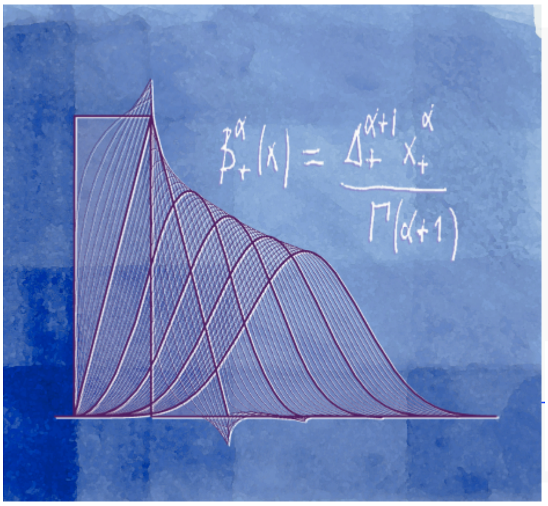
$\mathrm{L} = \mathrm{D}^{\gamma}$ (fractional derivative) Fourier multiplier: $(\mathrm{j}\omega)^{\gamma}$

$\mathrm{L_d} = \Delta_+^{\gamma}$ (fractional differences) Fourier multiplier: $(1 - \mathrm{e}^{-\mathrm{j}\omega})^{\gamma}$

$$\beta_+^0(r) = \Delta_+ r_+^0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1 - \mathrm{e}^{-\mathrm{j}\omega}}{\mathrm{j}\omega}$$

$$\vdots \qquad\qquad \vdots$$

$$\beta_+^{\alpha}(r) = \frac{\Delta_+^{\alpha+1} r_+^{\alpha}}{\Gamma(\alpha+1)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \left(\frac{1 - \mathrm{e}^{-\mathrm{j}\omega}}{\mathrm{j}\omega}\right)^{\alpha+1}$$

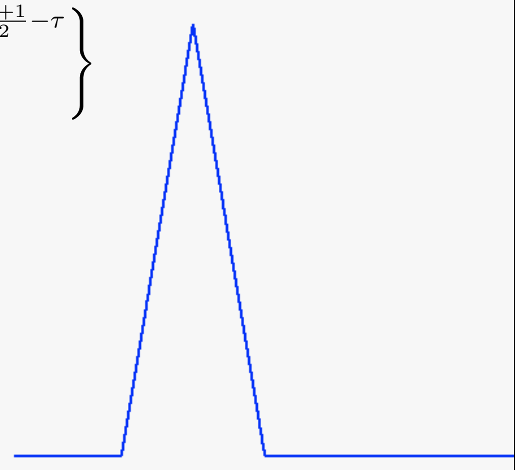


One-sided power function: $r_+^{\alpha} = \begin{cases} r^{\alpha}, & r \geq 0 \\ 0, & r < 0 \end{cases}$

Degree $\alpha = \gamma - 1$

Fractional (α,τ) B-spline

$$\beta_{\tau}^{\alpha}(t) = \mathcal{F}^{-1}\left\{ \left(\frac{1 - e^{-j\omega}}{j\omega}\right)^{\frac{\alpha+1}{2}+\tau} \left(\frac{1 - e^{j\omega}}{-j\omega}\right)^{\frac{\alpha+1}{2}-\tau} \right\}$$



B-spline of degree 1 and shift -1

- Asymptotic formula

$$\lim_{\alpha \to +\infty} \beta_{\tau}^{\alpha}(t) = \frac{1}{\sqrt{2\pi}\sigma_{\alpha}} \exp\left(-\frac{(t-\tau)^2}{2\sigma_{\alpha}^2}\right)$$

with $\sigma_{\alpha} = \sqrt{\frac{\alpha+1}{12}}$

Polyharmonic B-splines

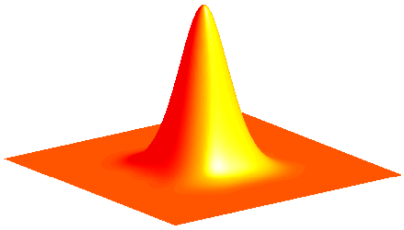
Fractional Laplacian: $(-\Delta)^{\frac{\gamma}{2}} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \|\boldsymbol{\omega}\|^{\gamma}$

Isotropic discrete Laplacian: $\frac{1}{6}\begin{pmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{pmatrix} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad V_2(\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}})$

- Polyharmonic B-splines (Rabut, 1992; Van De Ville 2005)

Discrete operator: localization filter $V_\gamma(\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}}) = V_2(\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}})^{\frac{\gamma}{2}}$

$$\frac{V_\gamma(\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}})}{\|\boldsymbol{\omega}\|^{\gamma}} \xrightarrow{\mathcal{F}^{-1}} \beta_\gamma(\boldsymbol{r})$$



Continuous-domain operator: $\widehat{L}(\boldsymbol{\omega})$